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Interference of directed paths

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Abstract. We study the problem of interference between directed paths in two dimensions, on a Euclidean square lattice, and on a hierarchical structure. The connection between this problem and directed polymers in a random medium at zero temperature is presented through a direct construction. This mapping leads to a quantitative description of the exponential vanishing of the amplitude as a function of distance travelled within the medium. A phase transition occuring for a small concentration of impurities, which has been reported in the past, is analysed using a direct numerical study. Although we cannot reach a definitive conclusion, the critical point seems to vanish in the thermodynamic limit. A renormalization argument is proposed to account for this effect. Finally, the same problem treated on a hierarchical lattice is shown to give rise to pathological behaviour for large disorder. For small disorder, the connection with directed polymers is recovered.

1. Introduction

The problem of the conformation and energy scaling of directed polymers in random media at zero temperature is a topic which has received considerable attention since an analytic solution was proposed by Kardar, Parisi and Zhang [1] for the two-dimensional case. The solution of [1] lies on a mapping onto a nonlinear Langevin equation with noise (hereafter referred to as the KPZ equation). Since then, a number of applications in various fields have been proposed to be described by such a model.

However, in most cases considered, the noise was real and positive. Extensions of the initial model to situations where the noise was complex have been studied in particular by Medina *et al* [2]. The motivation for such studies was the relevance of this problem for different applications. Examples include Aaronov–Bohm oscillations in hopping conductivity [3], tunnel hopping in disordered systems [4], interference of directed path in quantum localization problems [5], high-temperature limit for spin glasses [6], ...

Phase diagrams for the directed walks with complex random weights have been considered theoretically in the limit of a large space dimensions by Cook and Derrida on a Cayley tree [7] using the simple random sign case, and by Goldschmidt and Blum in the general case using a variational method expected to be exact in the limit of infinite space dimensions [8]. For positive random weights, the directed polymer problem on a hierarchical lattice has first been consider by Derrida and Griffiths [9], and more recently by Medina and Kardar [10].

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Zhang [11, 12] has proposed a different approach to the problem through a mapping onto a singular Langevin equation.

We study such a problem in the simple limit where the phases can only assume two values: 1 with probability $(1-\pi)$ and -1 with probability π. We propose different results on this problem including a direct mapping onto a Burgers equation studied in [1], a study of an eventual second-order phase transition reported initially in [3, 4] and discussed in [5, 13, 14], and which seems to vanish when the system size diverges as suggested in [5, 13–15], a renormalization argument which supports the previous conclusion, and a hierarchical lattice formulation which shows the relation of this problem with a random polymer case with real positive noise, and which exhibits a pathological behaviour when π is close to $\frac{1}{2}$.

2. Statement of the problem

Let us consider a square lattice whose bonds i are attributed random weights η_i from a bi-modal distribution: $\eta = 1$ with probability $1-\pi$ and $\eta = -1$ with probability π. To any path $\mathcal{P}$ we assign a phase $\phi(\mathcal{P})$ equal to the product of η along the path

$$\phi(\mathcal{P}) = \prod_{i \in \mathcal{P}} \eta_i \,. \tag{1}$$

Due to the particular choice of η, ϕ can only assume two values ± 1. The principal axes of the square lattice are oriented at an angle $\pi/4$ with respect to a coordinate system (x, y). From now on, we will only consider paths lying on the lattice which are directed along the y-axis. Figure 1 illustrates the geometry used in this study. From each site i in the network, located at a distance y from the x-axis, there are $N = 2^y$ different directed paths connecting i to the x axis. Among them, a number N_i^+ of paths are such that $\phi(\mathcal{P}) = 1$, and N_i^- have $\phi(\mathcal{P}) = -1$. We introduce the notation n_i^+ for the density of positive paths connecting i to the x-axis

$$n_i^+ = \frac{N_i^+}{N_i^+ + N_i^-} \tag{2}$$

and $\alpha_i = (n_i^+ - n_i^-)$.

The problem we address is to study the statistical properties of α values as a function of various parameters such as the distance y or the probability π.

Numerically, the computation of the α field in the lattice is straightforward. The directedness of the paths allows us to formulate the problem in a simple transfer-matrix way. Noting that each directed path from i to the x-axis has to go through one of the two neighbours of i, j and k, α_i has the simple form

$$\alpha_i = \frac{(\alpha_j \eta_j + \alpha_k \eta_k)}{2} \tag{3}$$

where η_j and η_k are the η values assigned to bonds $j - i$ and $k - i$, respectively.

The lattice used extends over a distance L_x along the x-axis and we implemented periodic boundary conditions. The length of the lattice along the y-axis is denoted L_y. The computation is initialized to $\alpha_i = 1$ along the x-axis and then α are computed along lines parallel to x using (3). Moreover, as we will see below, the average of $|\alpha|$ decreases exponentially with y. Thus, we rescaled all the values of α along each line so that the maximum value is equal to 1. The rescaling factor was recorded so as to revert to the actual value of α whenever needed. The rescaling procedure is simply to be considered here as a convenient way to preserve the desired accuracy with no consequence for the results.

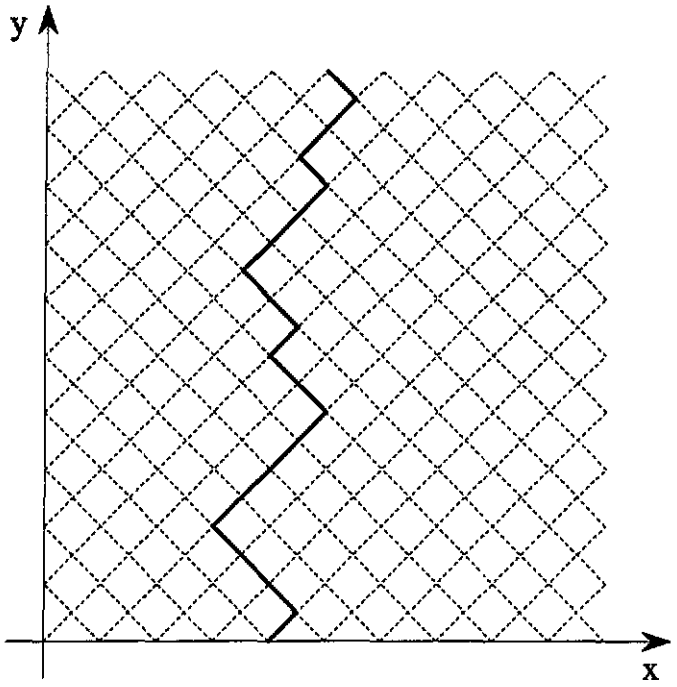


Figure 1. Starting from any site on the square lattice we consider the set of directed paths which end on the x-axis. Periodic boundary conditions are implemented in the x-direction.

3. Relation with KPZ equation

In order to establish the connection between the problem introduced in the previous section and the KPZ equation, we use a direct argument, different from the one used by Medina *et al* [2] who pointed out the analogy between the average value of the $2m$th power of $N\alpha$ and the statistics of m attracting particles in a one-dimensional medium.

Our starting point is (3). Due to the geometry of the lattice we use, we first iterate (3) twice so that we can relate the value of $\alpha(x, y+2)$ to $\alpha(x+a, y)$ with $a = -1$, 0 and 1. We can express such a dependence in the following form:

$$\alpha(x, y+2) - \alpha(x, y) = \tfrac{1}{4}(\alpha(x+1, y) - 2\alpha(x, y) + \alpha(x-1, y)) - (a_{-1}\alpha(x+1, y) + a_0\alpha(x, y) + a_1\alpha(x-1, y)) \,. \tag{4}$$

Only the three coefficients a_{-1}, a_0 and a_1 are dependent on the value of the η parameters in the six bonds which connect $(x, y+2)$ to the three sites $(x+a, y)$. a_{-1} and a_1 can take the values 0 and $\frac{1}{2}$ and a_0 can amount to 0, $\frac{1}{2}$ or 1. Equation (4) can be seen as the discretization of a continuous equation which can be written as

$$\frac{\partial\alpha(x, y)}{\partial y} = D\frac{\partial^2\alpha(x, y)}{\partial x^2} - A\alpha(x, y) \tag{5}$$

where $A \propto (a_{-1} + a_0 + a_1)$. We have neglected in this equation the non-local nature of the last term in the discretized form (4). In fact, we could also have made a Taylor expansion of α and thus added some additional contribution in $\partial\alpha/\partial x$. However, these terms should

not be determinant for the scaling properties of α. Indeed the scale of the non-locality is one lattice unit, and the diffusion term should dampen short-scale fluctuations of α. This last factor can be decomposed as the sum of a constant term plus a fluctuating part with zero mean: $A = \overline{A} + \delta A$. Therefore, equation (5) is a linear Burgers equation with a multiplicative noise

$$\frac{\partial \alpha(x, y)}{\partial y} = D\frac{\partial^2 \alpha(x, y)}{\partial x^2} - \overline{A}\,\alpha(x, y) - (\delta A)\alpha(x, y)\,. \tag{6}$$

Such an equation has been solved by Kardar, Parisi and Zhang [1] using a change of variable $\beta = \log(\alpha)$ so as to obtain a nonlinear diffusion equation with an additive noise

$$\frac{\partial \beta(x, y)}{\partial y} = -\overline{A} + D\frac{\partial^2 \beta(x, y)}{\partial x^2} + D\left(\frac{\partial \beta(x, y)}{\partial x}\right)^2 - \delta A\,. \tag{7}$$

It should be noted that using a similar starting point, but different approximations, Zhang [10] has proposed a mapping from the complex directed problem onto a different nonlinear Langevin equation with a singular term of the form $\log(|\partial\beta/\partial x|)$ in addition to (7).

Kardar, Parisi and Zhang [1] have shown that (7) gives rise to a non-trivial scaling property on the fluctuations of β as a function of y, i.e.

$$\langle \beta^2 \rangle - \langle \beta \rangle^2 = y^{2/3}\psi\left(y^{1/3}/L_x^{1/2}\right) \tag{8}$$

in two dimensions, where ψ is a scaling function which is constant for small argument ($\psi(x) \sim x^0$ for $x \ll 1$) and which scales as $\psi(x) \sim x^{-2}$ for $x \gg 1$. For the specific problem studied in this paper, Medina *et al* [2] have recovered essentially the same result using a different argument. The mapping onto a Langevin equation has initiated some debate still open in the literature [2, 9, 10]. The scaling form (8) can be rewritten introducing a correlation length which grows as $y^{2/3}$. This correlation length reflects the extent of transverse fluctuations of the paths. On the basis of numerical simulations and renormalization argument, Zhang [9] found a power-law increase of the correlation length with a different exponent $(\frac{3}{4})$. This result motivated his development of a new (singular) Langevin equation for describing such systems [10]. Gelfand [16] reported the result of numerical simulations in the field which favoured the results of Medina *et al* [2].

The constant $\overline{A}$ which appeared in (6) becomes simply a constant drift in (7) and thus can be eliminated by a suitable change of reference ($\beta \to \beta - \overline{A}y$). Referring to the definition of β, this leads to an exponential decay of the average value of $|\alpha| \propto e^{-y/\xi_\parallel}$. Let us now try to estimate $\xi_\parallel$. The constant $\overline{A}$ can be computed easily to amount to

$$\overline{A} = 4\pi(1 - \pi) \tag{9}$$

by taking into account all possible configurations of the six bonds involved in the derivation of (4) and setting $dy = 1$. In order to obtain an estimate of ξ we neglect the noise term in (4) and consider the case of an invariance of α along the x-axis (this form is the one for which the relaxation distance is the largest). Thus we look for a solution of (4) with no noise, using $\alpha = Ke^{-y/\xi_\parallel}$. Inserting this expression in (4), we obtain

$$e^{-2/\xi_\parallel} = 1 - \overline{A}\,. \tag{10}$$

Using the expression (9), we finally obtain

$$\xi_\parallel = -1/\log(|1 - 2\pi|)\,. \tag{11}$$

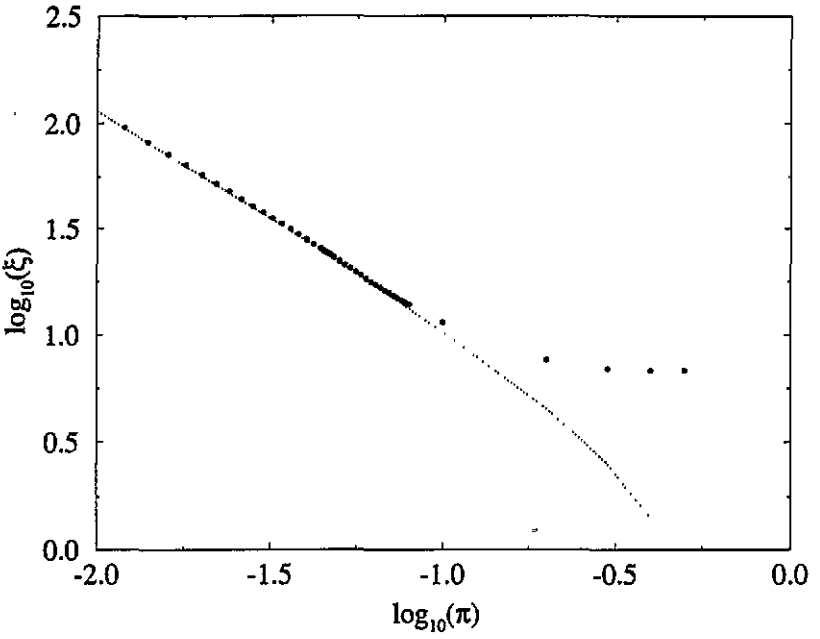


Figure 2. Numerical estimate of the relaxation scale $\xi_{\parallel}$ as a function of π. These data have been estimated on a Euclidean square lattice for a system of width $L_x = 500$ and length $L_y = 10^5$ for each point. The theoretical expectation (equation (11)) is shown as a dotted curve.

In particular, for small π, $\xi_{\parallel}$ *can be simplified to* $\xi_{\parallel} \approx 1/2\pi$.

Figure 2 shows the numerical estimate of $\xi_{\parallel}$ as a function of π together with (11). The agreement is good for small π values. For π close to $\frac{1}{2}$, taking into account the average value of A gives an estimate of $\xi_{\parallel}$ which is underestimated by a large amount. The reason for this deviation is to be looked for in the fact that the fluctuations in the noise dominate the average level. Indeed we have estimated the damping length considering a constant α, which is legitimate for small π, but which clearly breaks down for π close to $\frac{1}{2}$. We will come back on this point below.

As a direct consequence of this mapping, we check the scaling of the fluctuations of $\beta = \log(|\alpha|)$ with y. Figure 3 shows such an evolution for two different values of π: 0.5 and 0.05. The system size is $L_x = 10\,000$, and $L_y = 10\,000$ with an average over 1000 realizations. We do not see a simple power-law behaviour. Past an initial transient regime, the apparent exponent increases very slowly with the system size. The exponent b, such that

$$\langle \beta^2 \rangle - \langle \beta \rangle^2 \propto y^{2b} \tag{12}$$

when estimated for distances $y > 10^3$, gives $b \approx 0.31$, which is consistent with the expected value $b = \frac{1}{3}$ (equation (8)). A similar study on smaller system sizes has been reported by Medina *et al* [2] for $\pi = \frac{1}{2}$ and they also observed a slow approach to the $\frac{1}{3}$ exponent for $y > 10^3$. Our result is also consistent with the numerical result of Gelfand [16] on the same problem.

For $\pi = 0.05$, one obtains essentially the same result as for $\pi = 0.5$. The only major difference is the absence of a bump for small scales, which in any case is irrelevant for the

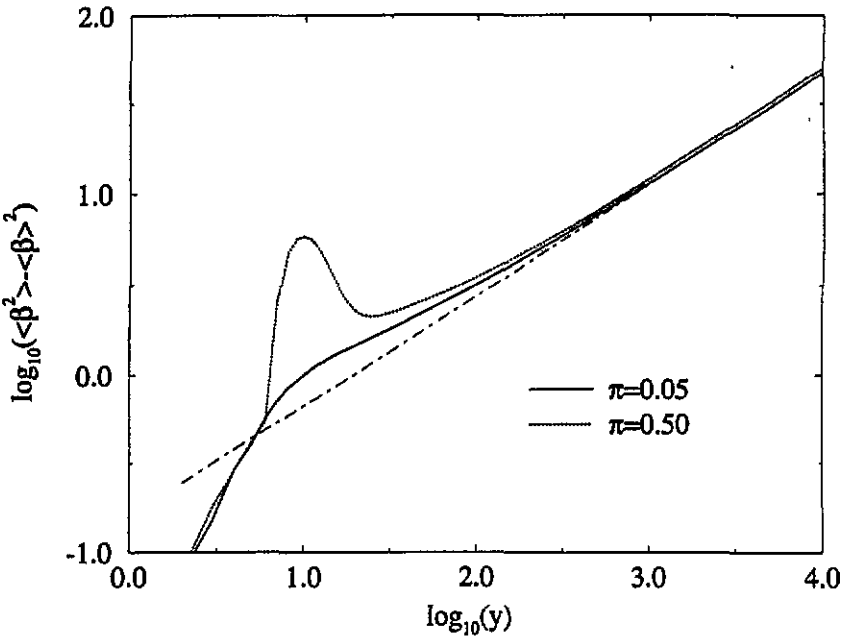


Figure 3. Log–log plot of the fluctuations $(\langle\beta^2\rangle - \langle\beta\rangle^2)$ of $\beta = \log(|\alpha|)$ versus y for $\pi = 0.5$ (dotted curve) and for $\pi = 0.05$ (full curve). A best fit of slope 0.63 is shown as a chain line.

asymptotic scaling. One surprising coincidence is to be noticed, however: the magnitude of the fluctuation is almost identical for these two values of π. We have no explanation to account for this observation.

4. Noise-free evolution

Starting from (4), let us consider the evolution of α in the absence of noise, i.e. turning the coefficients a_{-1}, a_0 and a_1 into their respective expectation values:

$$\alpha(x, y+2) - \alpha(x, y) = \tfrac{1}{4}\left(\alpha(x+1, y) - 2\alpha(x, y) + \alpha(x-1, y)\right) - \pi(1-\pi)\left(\alpha(x+1, y) + 2\alpha(x, y) + \alpha(x-1, y)\right). \tag{13}$$

Let us look for solutions of the latter equation as dampened exponentials of the form $\alpha(x, y) \propto \mathrm{e}^{-sy}\mathrm{e}^{ikx}$. Inserting this particular form in (13) gives

$$\mathrm{e}^{-2s} = \left(\tfrac{1}{2} - 2\pi(1-\pi)\right)(\cos(k) + 1). \tag{14}$$

We have already considered this equation in the case $k = 0$, so as to obtain the rate of decay of $|\alpha|$, in equation (10). From equation (14), we see that there always exists a real root s for all wavenumbers k. The critical damping is always reached for an α function which oscillates with a period of two lattice spacings. In the particular case $\pi = \frac{1}{2}$ all modes are at the critical damping. However, in this discussion, we only consider the noise-free case. In the original problem (4) fluctuations may thus induce either an exponential relaxation or a dampened oscillatory behaviour in y. Upon averaging along x or y (in a steady-state

regime) the dampened oscillatory behaviour yields naturally a zero-valued order parameter C, whereas an exponential relaxation leads to a non-zero value.

Moreover, upon a rescaling of the y-axis, all values of π apart from 0 and $\frac{1}{2}$ can be mapped onto each other, using the reduced variable $y/\xi_{\parallel}$. But in this transformation, one should note that the noise term is not invariant.

Let us also note one particular feature of (13) in the case where $\pi = \frac{1}{2}$. In the noise-free equation, there is a perfect cancelation of the RHS so that it reduces to

$$\alpha(x, y+2) = 0. \tag{15}$$

Thus, in this case, the noise is fundamental. Moreover, the coarse-grained local noise we have introduced in (5) becomes illegitimate, since it provides essentially the spatial coupling in the α field. Otherwise, we would expect independent random walk evolutions for each x, and thus $b = \frac{1}{2}$ in (12). Figure 6 shows a map of the spatial distribution of 'plus' and 'minus' sites for $\pi = \frac{1}{2}$. We indeed see on the figure that no more long-range structure can be found, in agreement with (15) which states that after a distance of 2 along the y-axis, no memory of the previous state is kept on average.

The above considerations are specific to the case $\pi = \frac{1}{2}$. As soon as $\pi \neq \frac{1}{2}$, equation (5) should hold, and thus we should recover the properties recalled above concerning the fluctuations of β. There may lie the reason why the approach to the expected b exponent is so slow.

5. Phase transition for low π

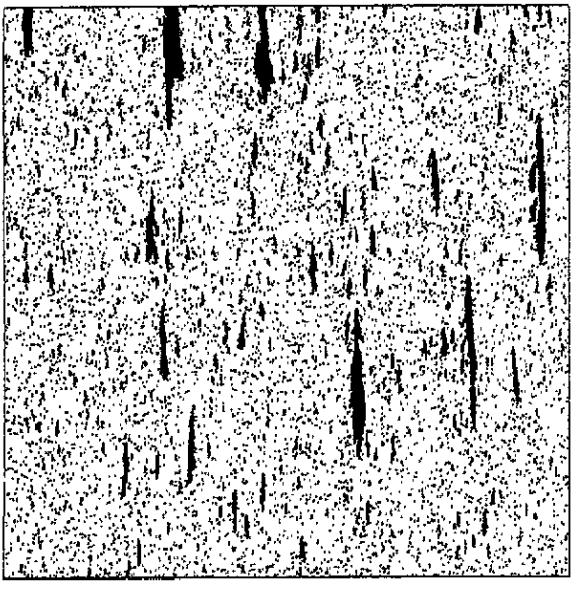
Apart from the scaling of $|\alpha|$ and its statistical fluctuations, another remarkable feature of this problem has been reported by Nguyen *et al* [3, 4]. These authors have observed that for large π, positive and negative α occured with the same probability, whereas for small π, α is mostly positive. A critical value π_c which separates these two regimes was estimated to be $\pi_c \approx 0.05$. Although the authors have proposed a rough argument which suggested a first-order phase transition, the numerical result they have obtained suggested that the transition was of second order. The argument proposed by Nguyen *et al* was discussed in details by Shapir and Wang [5] who suggested that the phase transition might simply be a slow cross-over which would vanish for large system size. In a later work [13], Shapir *et al* have performed a series expansion to analyse this transition. Although the system sizes considered were rather small, they did not observe such a transition. However, due to the limitation in size, they did not consider as reliable the data for $\pi < 0.10$. A small review of this question is given in [14]. A numerical study of this question has been performed by Medina and Kardar [15] and they found a very slow shift of the critical π_c toward 0 for large system sizes.

Let us note that for π larger than 0.5, the following transformation $\alpha \to (-1)^y \alpha$ maps the problem to $\pi \to (1-\pi)$. Therefore, we will restrict our study to the case $0 < \pi \leqslant 0.5$.

Figures 4(a) and 4(b) show the spatial distribution of sites for which $\alpha < 0$ in two 512×512 lattices for $\pi = 0.04$ and $\pi = 0.06$. Such pictures support qualitatively the occurrence of a transition. We will see, however, that a more quantitative study shed some doubts on this transition in the thermodynamic limit.

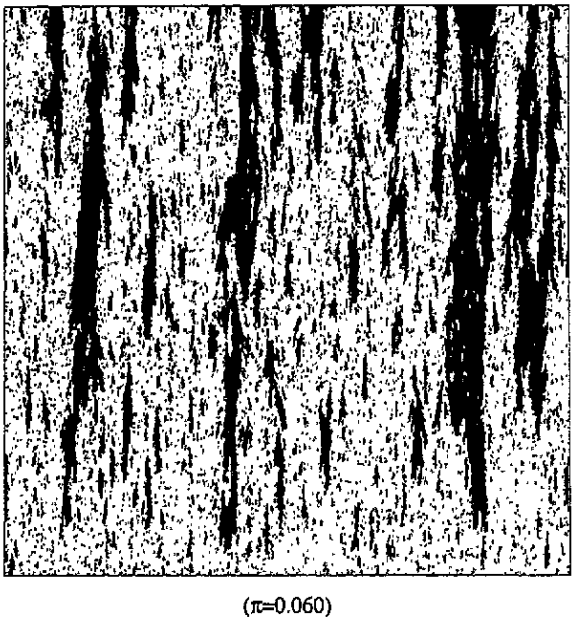
A convenient order parameter to study this transition is C defined from the distribution function $f(\alpha)$ of α

$$C = \left(\int_0^1 f(\alpha) \mathrm{d}\alpha - \int_{-1}^0 f(\alpha) \mathrm{d}\alpha \right)^2 . \tag{16}$$



(π=0.040)

(*a*)



(π=0.060)

(*b*)

Figure 4. Map of the sites i (shown in black) for which $\alpha_i < 0$ on a 512×512 lattice. Periodic b and c are implemented laterally as in figure 1. Values of π are respectively 0.04 and 0.06 in (*a*) and (*b*).

Figure 5 shows the evolution of $C(\pi)$ obtained numerically for small system sizes 100×100 such as those considered in [3, 4]. This graph is quite comparable to the one

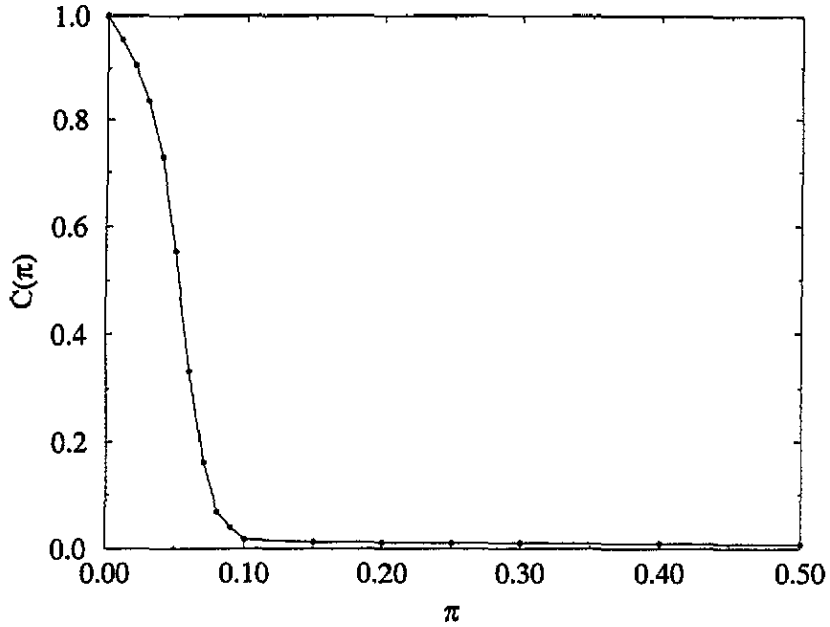


Figure 5. Evolution of $C(\pi)$ obtained from Monte Carlo simulation for small system sizes 100×100.

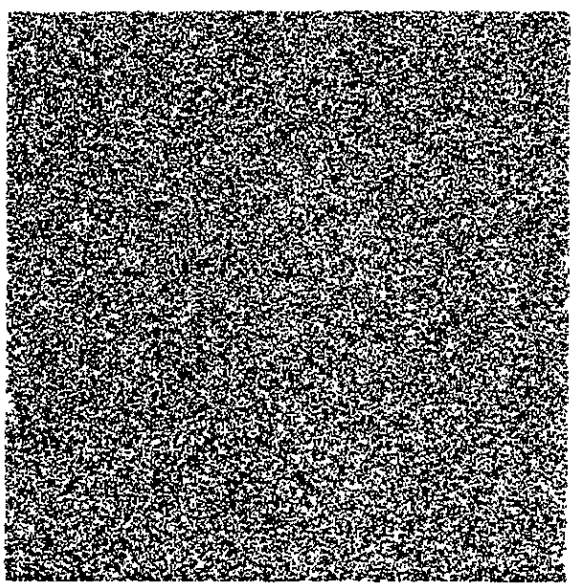


Figure 6. Same as figure 4, for $\pi = 0.5$. No more large-scale spatial structure can be seen in contrast with lower values of π.

obtained by Nguyen *et al*, and indeed suggests the occurrence of a second-order phase transition. A more extensive numerical study discussed below will however show that

the critical probability π_c seems to decrease systematically with the system size and may vanish in the thermodynamic limit, as suggested by Shapir and Wang [5]. The next section is devoted to the discussion of the numerical results.

6. Numerical results

We report in this section the results of numerical simulations concerning the transition for low π values.

In order to study the transition at low π values, we considered strip-like geometries. As a first attempt, we studied strip of moderate size along the x-axis (up to 500), and length along the y-axis (up to 5×10^5). Unexpectedly, this procedure gave rise to a very noisy behaviour of $C(\pi)$ in the interval $0.03 < \pi < 0.05$. A hint to understand this strange property is revealed in figure 7, where we have plotted the evolution of C estimated along rows parallel to the x-axis as a function of y. We see that at very large intervals, the system oscillates between two states (majority of + sites, and majority of − sites), with transient regimes where C is close to 0. The intervals at which the entire width of the system flips are very long and very irregular, so that it is very difficult to reach a reliable average data.

In order to circumvent such a difficulty, we consider another strip geometry where L_x is very large, and L_y moderate. However, in this geometry, the proximity of the initial line is crucial. We used this property in the following way: If the transition is of second order, with a singular behaviour of the order parameter as $C(\pi) \propto (\pi_c - \pi)^\beta$, and a divergence of the parallel correlation length as $\xi \propto |\pi_c - \pi|^{-\nu}$, at threshold, $\pi = \pi_c$, we expect that the average C with exhibit a finite-size effect. In fact we have used the notation C to note the dependence of the order parameter with π in an infinite system size. Right at threshold, the order parameter will depend explicitly on

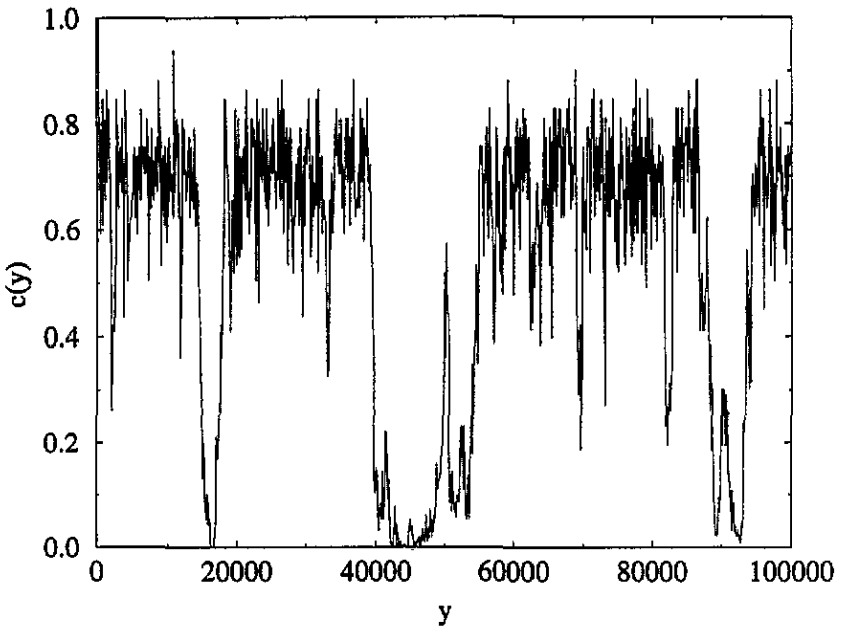


Figure 7. Evolution of the order parameter $c(y)$ as a function of y, for a strip-like geometry $L_x = 100$, $L_y = 50\,000$, and $\pi = 0.04$.

the system size as a power law. We note $c(y)$, the average of this order parameter as a function of the distance y for $\pi = \pi_c$. The power-law dependence can be written

$$c(y) \propto y^{-\beta/\nu}. \tag{17}$$

Away from the threshold, such power-law behaviour should be limited to an extent $y = \xi$, above which either $c(y)$ saturates to $C(\pi)$ for $\pi < \pi_c$, or $c(y)$ vanishes faster than any power law if $\pi > \pi_c$.

Figure 8 gives the evolution of $c(y)$ as a function of y for various trial values of π ranging from 0.036 to 0.046, and $L_x = 5 \times 10^5$, $L_y = 500$. We see on the figure that all the curves display a marked downward curvature at large y with a very slow decay at smaller y. For the smallest values of π, figure 8(b) shows the data with a greater magnification, which shows clearly an upward curvature for law y followed by a downward one at larger values. From the study of smaller system sizes, say $y < 100$, we may therefore have estimated the critical π_c at a value 0.040, but considering larger system sizes indicates $\pi_c < 0.036$. This continuous decrease of the estimate of π_c together with the presence of an inflexion point in the evolution of $c(y)$, and the very low value of the β/ν exponent of (17) suggest that it is possible that the transition occurs only at $\pi = 0$, for an infinite system size. Numerically, it is impossible to reach any definitive conclusion. Our only claim considering the numerical data will thus be $\pi_c < 0.036$ and $\beta/\nu < 0.04$. These results are fully consistent with the numerical results of Medina and Kardar [15] obtained on smaller system sizes.

In order to shed some light on this curious behaviour, we propose in the next section an approximate renormalization argument which suggests the possibility of $\pi_c = 0$.

7. An approximate renormalization

We introduce an approximate computation which suggests that there might be only one single stable fixed point $= \frac{1}{2}$ for $0 < \pi < 1$.

Let us consider the distribution, f, of α. We introduce the probability p^+ that α is positive, i.e. $p^+ = \int_0^1 f(\alpha)\mathrm{d}\alpha$, and similarly, $p^- = 1 - p^+$. The simplifying assumption is to state that, apart from a scaling factor, the distribution of negative α is identical to the one of the positive α. More precisely, for $\alpha > 0$, we assume $f(-\alpha) = (p^-/p^+)f(\alpha)$. Furthermore, we will neglect correlations in the values of α for neighbouring sites.

Using these hypotheses, it is simple to carry out the computation of $p^+(y+1)$, as a function of $p^+(y)$ using (3). The probability that α is positive at a distance $y+1$ is obtained by considering all possible η values of the bonds connecting the site to the two neighbours at distance y. For a given configuration the α value at the inspected site is written as the sum or the difference of the two α values at the y level. Making use of the assumption $f(-\alpha) = (p^-/p^+)f(\alpha)$, it is simple to express the probability that the resulting sum or difference is positive. In order to do so, we consider all possible combinations of α sign, whose probabilities are straightforwardly computed, and thus we are left with computing the probability that the sum or the difference of two *positive* variables distributed along the same law is positive. The sum is trivially always positive, and the difference is positive with probability $\frac{1}{2}$. Finally, we obtain

$$\begin{aligned} p'^+ = (1-\pi)^2\left(p^{+2} + p^+(1-p^+)\right) + 2\pi(1-\pi)\left(p^+(1-p^+) + \tfrac{1}{2}p^{+2} + \tfrac{1}{2}(1-p^+)^2\right) \\ + (\pi)^2\left(p^+(1-p^+) + p^+(1-p^+)\right). \end{aligned} \tag{18}$$

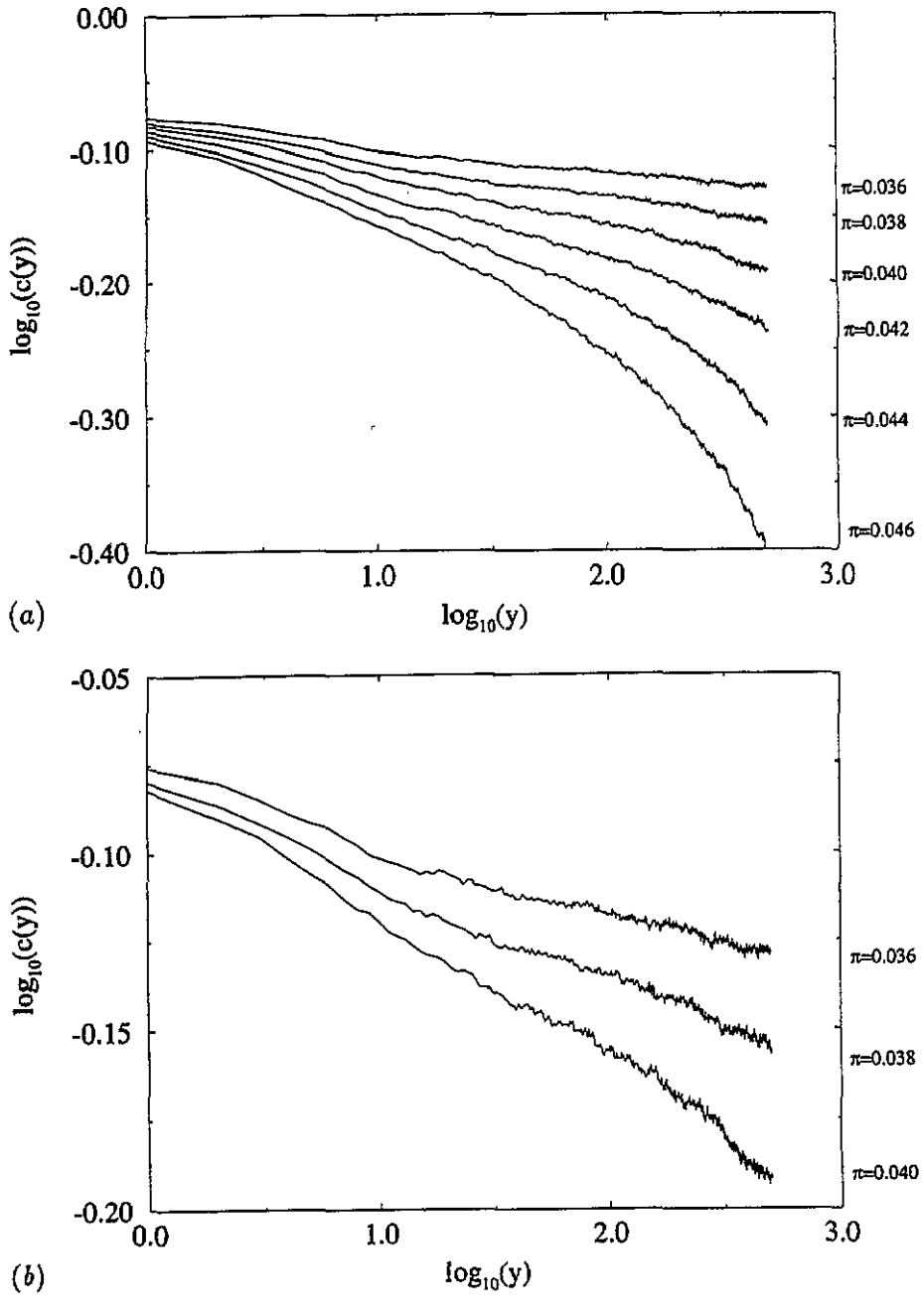


Figure 8. (*a*) Transfer-matrix computation of $c(y)$ as a function of y for various values of π. The geometry is strip-like $L_x = 5 \times 10^5$, and $L_y = 500$. (*b*) Same as (*a*) for the smaller values of π with a magnified c scale.

Rearranging this sum, the recursion formula reduces to the simple linear expression

$$p'^{+} = \pi + p^{+}(1 - 2\pi)\,. \tag{19}$$

From this simple recursion, we see that as soon as π is less than 1, the only fixed point is

$p^+ = \frac{1}{2}$. However, the time needed to reach this fixed point diverges as π approaches 0. In addition, this time may also be underestimated in this computation due to the presence of spatial correlations at low π.

The weakness of this model is mostly apparent for $\pi = 0$. In this case, starting from a random distribution of $\alpha(x)$, the system will evolve to a constant value $\alpha(x) = \alpha_0$, where α_0 is nothing but the arithmetic average of the initial $\alpha(x)$, and thus after a transient regime, p^+ will be either 1 or 0. However, our renormalization gives any p^+ as being a fixed point in the transformation. One can see in this particular case that the starting hypothesis—similarity of f for positive and negative α—is in default. It is, unfortunately, not possible to weaken this hypothesis and still preserve the possibility of carrying over the computation of p without specifying more precisely the distribution f.

8. Hierarchical lattice

Let us now introduce another approach still in the spirit of a real-space renormalization, but *using a different lattice.*

A hierarchical lattice consists of iterating a simple transformation which turns one bond into a small lattice. Upon iteration each bond of the basic lattice is again turned into a small lattice itself. We have chosen here as a small lattice four bonds assembled in a diamond shape. With such a transformation, the resulting lattice has a dimension equal to two.

The simplicity of the geometrical construction permits us also to deal with effective properties of the system in a simple recursive way, and thus to obtain exact renormalization equations.

The construction of the basic lattice can be decomposed in two steps. First we assemble four bonds two by two in series, and then the resulting 'macro-bonds' are connected in parallel. It suffices to write down the rules for combining two lattices in series and in parallel to obtain the complete renormalization equations. Let $n_{i\pm}$ for $i = 1, 2$ be the fraction of positive and negative paths in two lattices. If we connect the two lattices in series, the resulting $n'_\pm$ is simply

$$n'^+ = n_1^+ n_2^+ + n_1^- n_2^- \qquad n'^- = n_1^+ n_2^- + n_1^- n_2^+ . \tag{20}$$

If we connect two lattices in parallel, then the transformation reads

$$n'^+ = \tfrac{1}{2}(n_1^+ + n_2^+) \qquad n'^- = \tfrac{1}{2}(n_1^- + n_2^-) . \tag{21}$$

From those two transformations it is easy to compute the transformation of $\alpha = (n_+ - n_-)$:

$$\begin{aligned} \alpha' &= \alpha_1 \alpha_2 \qquad \text{for series} \\ \alpha' &= \tfrac{1}{2}(\alpha_1 + \alpha_2) \qquad \text{for parallel} . \end{aligned} \tag{22}$$

If the α were only positive numbers, then we could map exactly this problem onto one equivalent to the KPZ equation. We introduce $\beta = \log(\alpha)$. The series transformation (22) becomes $\beta' = \beta_1 + \beta_2$, while the parallel transformation can be written

$$\beta' = \log(e^{\beta_1} + e^{\beta_2}) - \log(2) . \tag{23}$$

As the distribution of $\beta = \log(\alpha)$ gets broader while the system size increases, the above equation can be simplified to

$$\beta' = \max(\beta_1, \beta_2) - \log(2). \tag{24}$$

The addition of the constant term in the latter equation can be interpreted as a steady drift term which can be subtracted off through a redefinition of $\beta \to \beta + n\log(2)$, where n is the generation. This does not affect the scaling properties of the fluctuations of β. The series and parallel transformation thus becomes simply a sum and a maximum operation on β. This is precisely the way one usually models the distribution of energy of a directed polymer in a disordered media at zero temperature on a hierarchical lattice (see, for example, [9, 17]). Therefore one recovers essentially the results introduced in section 3, using a very different route.

However, the above argument cannot be simply obtained in the case where the α can assume positive or negative values. The problem does not come from taking the logarithm of α. This can indeed be adjusted by considering the distribution of positive α and negative α separately. In particular for the case $\pi = 0.5$ these two distributions are identical, and thus the only modification is in (23) where one may have to consider the case of a minus sign between the two exponentials. This again can be simplified by factorizing out the exponential of the largest β in the argument of the logarithm and thus one gets the result (24) again.

The main limitation comes from a pathology of the hierarchical lattice. Considering (22), there is a finite probability to encounter $\alpha = 0$. For small lattices ths probability is large, and, in fact, it very quickly converges to one. Let us introduce $f(\alpha)$, the statistical distribution of α, and $q = f(0)$. The probability that after a series transformation $\alpha = 0$ is simply $q' = 2q - q^2$. For a parallel transformation, $q' = \int f(\alpha) f(-\alpha) \mathrm{d}\alpha$. Extracting the

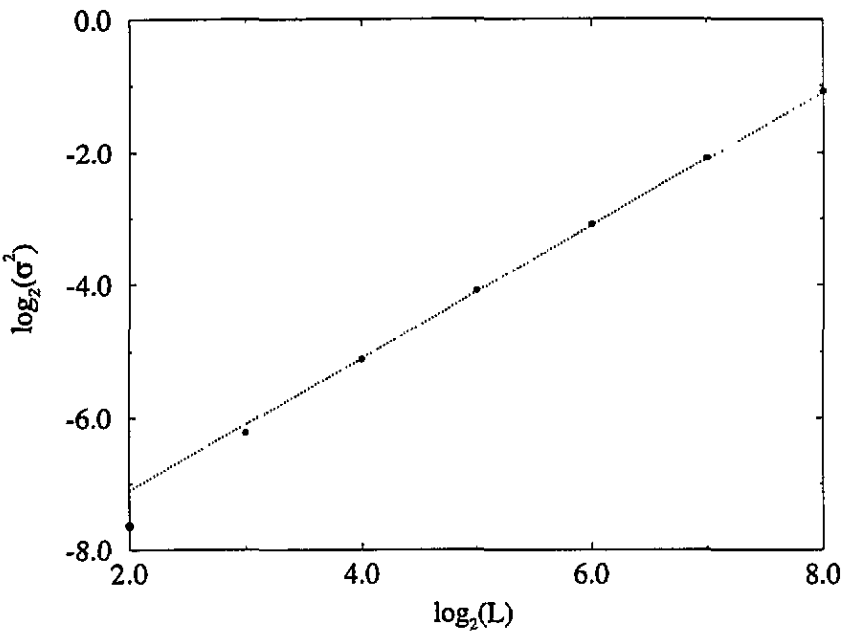


Figure 9. Variance of the distribution of β, $\sigma^2 = \langle\beta^2\rangle - \langle\beta\rangle^2$, on the hierarchical lattice as a function of the system size $L = 2^n$, where n is the generation of the lattice. The dotted line has a slope $2b = 1$.

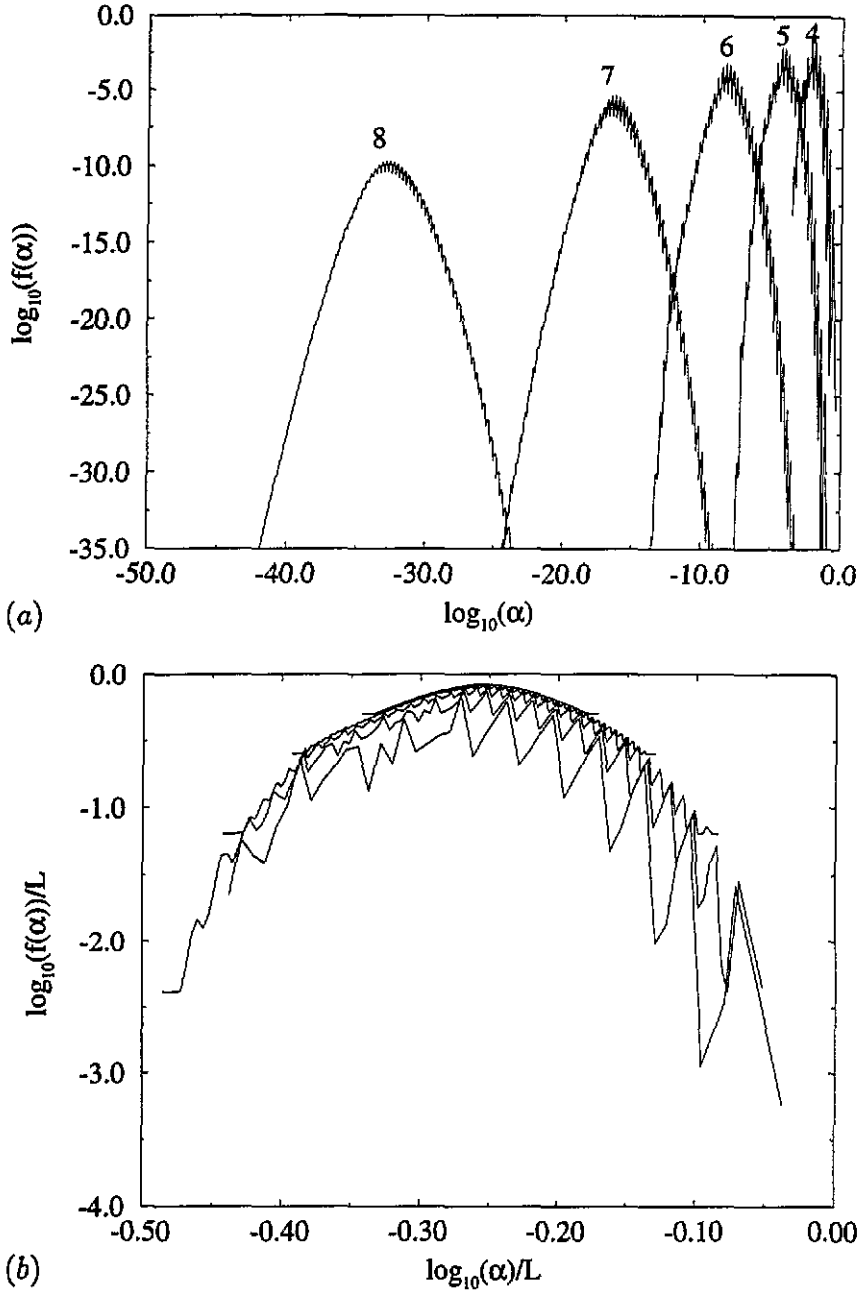


Figure 10. (*a*) Distributions f of the positive α, for various generations $n = 4$ to 8 on the hierarchical lattice, using $\pi = 0.5$. (*b*) Rescaled distributions $\log(f)/L$ versus $\log(\alpha)/L$, for the five sizes considered in figure 10(*a*).

value $\alpha = 0$ from the latter integral, we obtain a simple lower bound $q' > q^2$. Therefore, after one generation, q is turned into q'' which is bounded by $q'' > q^2(2-q)^2$. Assuming that q is large enough after a few generations (what is observed numerically) then q converges to 1 faster than an exponential. An exponential convergence is obtained in the case of an

equality instead of an inequality.

This dominant role played by $\alpha = 0$ has the major consequence of controlling the scaling of the variance of $\log(|\alpha|)$, as well as the shape of the distribution f itself. Considering the distribution of positive α, we see that the series transformation is reduced to the multiplication of two α's, or the sum of two β's, whereas the parallel transformation on the same part of the distribution is dominated by the addition of one α with 0, divided by two. Thus the parallel transformation is a mere translation by a factor of $\log(2)$, whereas the series transformation gives rise to a normal distribution of β or log–normal distribution of α as a consequence of the central limit theorem. As a result of this degeneracy, the scaling of the variance of β is that of the sum of independent random variables, i.e. $b = \frac{1}{2}$ in the notations of (12). Figure 9 shows such an evolution for $\pi = \frac{1}{2}$ from direct numerical simulation of the distributions f with the lattice generation (up to the 8th generation). In figure 10(a), we show the distribution of the positive α in a log–log plot, for different generations from 4 to 8. A rescaling of the distribution by $1/L$ gives the Gaussian limit (independent of L) expected from the above argument. The rescaling shown in figure 10(b) is in good agreement with this conclusion.

For small values of π, there seems to be a clear sign of transition to a regime where all α have the same sign, and where simultaneously, $f(0)$ converges to zero. This regime now simply gives rise to the scaling of the directed polymers as shown above. However, this transition is also very sensitive to the particular geometry of the hierarchical lattice, because of the particular role played by $\alpha = 0$. We have not investigated in detail the sensitivity of this transition with respect to the numerical discretization of the distribution f, which is a necessary step in the numerical simulation of this problem.

9. Conclusions

We have studied the problem of the interference of directed path on a disordered lattice with a simple bi-modal distribution of bonds ($\eta = \pm 1$). We have shown in a direct way the connection of this problem with the Kardar–Parisi–Zhang equation, as first suggested by Medina *et al* [2]. We have discussed the transition to a majority of paths sharing the same sign, and suggested that this transition may vanish in the thermodynamic limit as initially suggested in [5, 13, 14]. We have also shown that the same problem studied on a hierarchical lattice gives rise to a pathological behaviour for intermediate π values, due to the dominant role of an exactly balanced number of positive and negative paths. However, for low π values, a mapping onto a directed polymer in random media at zero temperature has been obtained. This problem belongs to the same class of universality as the Kardar–Parisi–Zhang equation, and thus it provides an independent confirmation of the above-mentioned result.

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